# Natural Number System and Its Fundamental Relationships

**(Proof of Fermat's Last Theorem)**

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Abstract:- Pierre de Fermat in 17th century wrote as a marginal note (later published by his son Samuel de Fermat) on the Diophantus' book Arithmetica ( Latin translation with commentary of the Greek book by Claude Gaspard de Bachet), while studying the natural number solutions of equation $x^2 + y^2 = z^2$, that, " No cubes of natural numbers can be split in to two cubes or a biquadrate can be split into two biquadrates or no other higher order greater than 2 of a natural number can be split in to the sum of two natural numbers having the same order, in which I have found a marvellous demonstration that this margin is narrow to contain." Ironically, Fermat didn't give the proof for this proposition during his life time. It was the last one to be proved of Fermat's propositions and hence historically called as Fermat's Last Theorem. In modern terms the theorem can be stated as "$x^n + y^n = z^n$ has no solutions for $n > 2$ in natural numbers." Mathematicians tried for centuries but could not construct the proof for general case. In 1994 Prof. Andrew Wiles (with the help of Richard Taylor) published a proof using the advanced ideas and techniques of mathematics and is significantly long and deep. But the attempt here is to understand the fundamental relationships of natural number system, the linear and trilinear (triangle inequality) relationships in which the number system manifests its significance in physical world since the time of early civilizations. Fermat's Last Theorem is here by demonstrated as a statement about the uniqueness of the trilinear relationship in natural number system by establishing the unique and comprehensive correlation between Euclidean geometry and natural number system (geometric algebra of natural number system). Both of them in conjunction demonstrates the principle of true model of relational dominance of trilinear relationships in natural number system, in which the triangle law of addition of physical quantities and the trigonometry of the space fundamentally depends upon.

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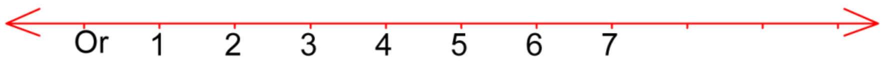
**1. Natural numbers.**

Natural numbers are represented as N=1,2,3,4,5…… As we all know that natural numbers are counting numbers and they are fundamentally used to count objects in nature.

**2. Natural numbers can be represented on a Euclidean line in order with some scale unit for 1.**




Or – Origin

As we represent the counting numbers or natural numbers on a Euclidean line in order with some scale unit for 1, the property of the natural numbers dramatically changes. Now the line with numbers represented on it (the number line) looks like a marked straight edge and the combination of numbers and the line or part of line becomes suitable as a system for making lengthwise (geometric) measurements.

Also, another important fact is that the natural numbers now form the part of the line, and the numbers can be represented as line segments of corresponding length (or a number corresponds to the specified magnitude of line segment with which the number is represented on a number line in relative to a scale unit for 1) and are bound to obey the relationships that can be developed for the line or part of the line (line segments) in a logical manner.

Therefore, the statement, "Natural numbers can be represented on a Euclidean line in order with some scale unit for 1," may be treated as an axiom and be called as the "linearity axiom" of natural numbers.

**3. Linear relationship between three numbers is a fundamental relationship in natural number system.**

Linear relationships between three natural numbers can be expressed as follows.

1) $x + y = z$

2) $x - y = z$ $(x > y)$

3) $x \times y = z$

4) $x \div y = z$ (y is a factor of x)

The second, third and fourth relations as shown above are the manifestation of the fundamental linear relationship $x + y = z$. They are called linear relationships or linear operations because all the above operations can be demonstrated as linear transformations on a number line. In the first case of addition of two numbers, the line segments representing the corresponding numbers to be added are combined together to get the line segment representing the resultant number. And the second case, to subtract a number from another, the line segment representing the former is deducted from that representing the latter and the remaining line segment represents the resultant number. The third case of multiplication of two numbers can be demonstrated as the recurrent addition of the line segment representing either of the two numbers, and the remaining number represents the number of recurrences its counterpart has to involve in the operation of addition to give the line segment representing the resultant number. The fourth case of division of two numbers can be demonstrated as the recurrent elimination of the line segment representing the dividend by the line segment representing the divisor and the number of recurrences needed for the total elimination of the dividend is the resultant number.

**4. Trilinear relationship – another fundamental relationship in natural number system.**

**4.1. The trilinear relationship between three natural numbers.**

Trilinear relationship is also a fundamental relationship between three natural numbers which is well known as the triangle inequality relationship in geometry. The term trilinear is used intentionally to specify that the relationship is fundamental in natural number system and in natural number system there is no notion of angles.

If three natural numbers x, y and z hold the following relationship between them, then a trilinear inequality relationship is said to exist between them.

$$\left.\begin{array}{l} x < z + y \\ y < z + x \\ z < x + y \end{array}\right\} \text{Each number of the triple is less than sum of the other two.}$$

**4.2. Lemma 1.**

The complete expression of trilinear relationship for three natural numbers can be stated that, for $z \geq y \geq x$, if $z < x + y$, there exists a trilinear relationship between them.

<u>Proof</u>

The logically possible relation between any three natural numbers is one of the following.

1) Three numbers are same.

$z = y = x$

2) Two numbers same and greater than the third.

$z = y > x$

3) One number greater than the other two equal numbers.

$z > y = x$

4) All the numbers different.

$z > y > x$

Here z may be taken as the large number in all the four relations. The large number reference does not indicate that there is a specific large number, but only to the number which can be considered as large in the triple. For the numbers x, y, z, if all are equal, then the large number is the number itself. It can be easily seen if the large number is less than the sum of the other two numbers, other two of the trilinear inequality naturally follow.

We can see that in the first two cases, $z = y = x$ and $z = y > x$, the large number z is always less than the sum of the other two numbers and there always exists a trilinear relationship between them. In the other two cases i.e. $z > y = x$ and $z > y > x$, a trilinear relationship exists between three numbers if and only if, the large number is less than the sum of the other two numbers.

Therefore, the complete expression of trilinear relationship in natural number system can be stated as, for three natural numbers $z \geq y \geq x$, if $z < x + y$, there exists a trilinear relationship between them.

**4.3. Lemma 2.**

The number of trilinears of the form $z < x + y$ exists for a natural number y with x such that $y \geq x$ and z the large number is $y(y + 1)/2$ and the set of triples for each y is unique.

Proof

The triples formed for y with the given condition $z \geq y \geq x$, $z < x + y$ are as follows.

Case (1), $(z = y = x)$, $\therefore y = y = y, y < y + y$ , there exists 1 trilinear triple.

Case (2), $(z = y > x)$, $\therefore y = y > x, y < y + x$, x can be 1,2,...,y-1, $\therefore$ y-1 trilinear triples.

Case (3), $(z > y = x)$, In this case $x = y$, but $z < x + y$, $\therefore$ z can take a minimum value of y+1 and maximum value of 2y-1, $(2y\text{-}1 < y + y)$.

$\therefore$ number of trilinear triples = (2y-1) - (y+1) +1=y-1.

Case (4), $(z > y > x)$, in this case x can be 1,2,.......,y-1,but $z < x + y$,$\therefore$ for x=1,

there will not be any trilinear as there would not be any $z < y+1$, between y and y+1.

Similarly, for x=2, there would be 1 trilinear and so on. ∴ for x = y-1, there would be y-2 trilinear triples.

∴ total number of triples for case (4) = 1+2+3+..........+y-2= (y-2)(y-1)/2= $(y^2 -3y +2)/2$.

The total of all the triples for the four cases = (1+ y-1 +y-1 +$(y^2 -3y +2)/2$) = $(y^2+ y)/2$= y(y+1)/2.

<u>To show the trilinear triples are unique for each y.</u>

Let the triples are formed for the 4 cases as shown above for the natural numbers $y_1$ and y such that $y_1 > y$. For case (1), the triple consists of only $y_1$ and y in each case and in the next two cases, each of them contains two $y_1$ or y accordingly. Let the triples for y by case (4) are of the form z > y > x., and for $y_1$ as $z_1 > y_1 > x_1$ (z, x and $z_1$, $x_1$ take suitable values in respective cases). Therefore it can be seen that any of the triples formed by cases (1) and (4) of $y_1$ and y respectively cannot be identical with any of them of cases (1) to (3) of y and $y_1$ respectively. Comparing cases (2) and (3) of $y_1$ and y together, as each of them contains two $y_1$ or y accordingly, whatever be the third one in both of them, they cannot be identical. As the triples for y by case (4) are of the form z > y > x and for $y_1$ as $z_1 > y_1 > x_1$ and since. $y_1 > y$, y can take only the position of $x_1$ in the triple formed by $y_1$. Now, if any of the triples formed by y and $y_1$ by case (4) has to be same, then either $z_1$ or $y_1$ has to take one of the x values of the triples formed by y. This is not possible as y > x and $z_1$, $y_1 > y$. Therefore, none of the triples formed by y and $y_1$ for case (4) is identical. Hence the set of triples formed for each y is unique.

<u>Let us demonstrate the above lemma with an example.</u>

Consider the case y=4, then x=1, 2, 3, 4.

The trilinear relations for y=4 are shown below.

Case (1), (z = y = x), i.e. the trilinear is 4, 4, 4.

Case (2), (z = y > x), x=1, 2, 3. y – 1 = 4 – 1=3 triples.

The trilinear triples are (1) 4, 4, 1(2) 4, 4, 2 (3) 4, 4, 3.

Case (3), (z > y = x), x = 4.y – 1 = 4 – 1=3 triples.

The trilinear triples are (1) 7,4,4 (2) 6, 4, 4 (3) 5, 4, 4.

Case (4), (z > y > x), x=1,2,3.

There will be no triple possible with x=1.

The triples are (1) 5, 4,2 (2) 5,4,3 (3) 6,4,3.

i.e. $(y^2 - 3y + 2)/2 = (4^2 - 3 \times 4 + 2)/2 = 3$.

The total number of triples, 4 cases, is 3+3+3+1=10.

$y(y+1)/2 = 4(4+1)/2 = 10$.

In fact, for each y, we get a unique set of triples as per the lemma above and therefore there exists infinite number of trilinear triples in natural number system and they are countable also.

**5. The natural numbers x, y and z holding the relationship $x^2 + y^2 = z^2$ represents a class of trilinears in natural number system.**

**5.1. The relationship $x^2 + y^2 = z^2$ in the natural number system.**

The relationship $x^2 + y^2 = z^2$, in which there are large number of natural number triples holding the relation was known to be understood by human civilisation before 1600BC. The solution to the equation was given in 3rd century AD by Diophantus of Alexandria, in Book II of his Arithmetica, and a more geometric version can be found in Book X of Euclid's Elements. The solution for $x^2 + y^2 = z^2$ is given as $x = u^2 - v^2$, $y = 2uv$ and $z = u^2 + v^2$, where u and v are any two natural numbers such that $u > v$. The relationship $(u^2 - v^2)^2 + (2uv)^2 = (u^2 + v^2)^2$ always holds true and it is an identity in the system of natural numbers.

As any pair of natural numbers of the form $u > v$ can satisfy $x^2 + y^2 = z^2$, if x is arranged as $u^2 - v^2$, y as 2uv and z as $u^2 + v^2$, shows the relationship is well spread throughout the natural number system. In fact, all the solutions to $x^2 + y^2 = z^2$ can be given as $x = m(u^2 - v^2)$, $y = 2muv$ and $z = m(u^2 + v^2)$ (x, y possibly transposed, $u > v$, u,v coprime and of opposite parity) and m, any natural number, is a proven fact.

**5.2. Lemma 3.**

The relationship $x^2 + y^2 = z^2$ represents an equality relationship of a class of triples holding trilinear inequality in natural number system.

<u>Proof.</u>

The numbers holding the relation $x^2 + y^2 = z^2$ forms a distinct class in natural number system is described in 5.1.

Let $x^2 + y^2 = z^2$, then $z^2 > y^2$, $z^2 > x^2$, $\therefore z > y$, $z > x$, x may or may not be equal to y does not affect the proof, but Fermat had proved that there are no isosceles triples for this case and the solution will be of the form $z > y > x$. Now $x^2 + y^2 < (x + y)^2$, as $(x + y)^2$ contains an additive term other than $x^2 + y^2 \Rightarrow z^2 < (x + y)^2 \Rightarrow z < (x + y)$.

Since $z < x + y$ and $z > y > x$, the other two inequalities $x < z + y$ and $y < z + x$ naturally follow and there exists a trilinear inequality in between them. .

The solution to the equation $x^2 + y^2 = z^2$, given as, $x= u^2 - v^2$, $y = 2uv$ and $z=u^2 + v^2$, fundamentally develops as an inequality from $u > v$ as $u^2 - v^2 + 2uv > u^2 + v^2$, $(z < x + y)$ and if $z > y,x$ , they also hold a trilinear inequality relationship, is demonstrated as follows.

$u > v \Rightarrow 2uv > 2v^2 \Rightarrow 2uv - v^2 > v^2$, adding $u^2$ on both sides $\Rightarrow u^2 - v^2 + 2uv > u^2 + v^2$ $\Rightarrow z < x + y$. In this $u^2 - v^2 < u^2 + v^2$ and to show $2uv < u^2 + v^2$, put $u = v + p$, p a natural number. $2uv \Rightarrow 2 (v+ p) v = 2v^2 + 2vp$. $u^2 + v^2 \Rightarrow (v + p)^2 + v^2 = 2v^2 + 2vp + p^2$.

$\therefore 2uv < u^2 + v^2$ for all p and $u^2 + v^2 > u^2 - v^2, 2uv$, $(z > y, x)$ and as $z < x + y$, the other two inequalities naturally follow and there exists a trilinear inequality in between them.

[The relationship $x^2 + y^2 = z^2$ represents a class of trilinear relationship and therefore a unique one in natural number system and may be considered as equivalent to the fundamental linear relationship $x + y = z$. It is important here to note that the equality $x^2 + y^2 = z^2$ is not obtained from the inequality relationship between them but we are only showing the otherwise that $x^2 + y^2 = z^2$ is holding a trilinear inequality relationship.]

**6. The equality relationship of all the trilinears in natural number system.**

As we can see that the relationship $x^2 + y^2 = z^2$ and its solution as $x = u^2 - v^2$, $y = 2uv$ and $z = u^2 + v^2$, represents a class of trilinears in the natural number system. It can be seen that there exists lot more trilinear triples not holding the relationship $x^2 + y^2 = z^2$ in natural system by lemma1. The number of triples holding a trilinear relation as per case (4) lemma1 for a natural number y is $(y^2 -3y +2)/2$ in which a relation of the form $x^2 + y^2 = z^2$ exists (as isosceles triples do not form this relation). And in many cases of y, there are no triples holding the relation $x^2 + y^2 = z^2$. As an example, for y=5, the number of trilinear triples exist as per case (4) lemma1 is 6 and none of these is of the form $x^2 + y^2 = z^2$. Hence there are infinite number of trilinear triples which cannot be expressed as $x^2 + y^2= z^2$. As in the case of 10, 11, 12 where $x = 10$, $y = 11$, $z = 12$, $z^2$ is less than $x^2 + y^2$, and for 10, 11, 15 where $x = 10$, $y = 11$, $z = 15$, $z^2$ is greater than $x^2 + y^2$. All the trilinear triples formed by the relation $z \geq y \geq x$ and $z < x + y$, excluding those satisfying $x^2 + y^2=z^2$, may be tentatively included in either of the following class $x^2 + y^2 < z^2$ or $x^2 + y^2 > z^2$.

∵ It is not at all surprising that as per the linearity axiom natural numbers can be represented as line segments, the inequality to equality transformation of all the trilinears is not established by algebra, as the general theory of straight-line segments holding triangle inequality relationship is Euclidean geometry.

## 7. The Euclidean geometry and natural number system.

### 7.1. Outline of Euclidean geometry.

Euclid has given his geometry in the book Elements which comprises of thirteen books (includes number theory and geometry of solids) and the Book1 deals with plane geometry. Euclidean geometry is built up of definitions of general terms occurring in geometry such as points, lines, surfaces, plane angles, circles, triangles (trilateral figures) etc. and the five postulates and the common notions which are logical statements in mathematics such as "Things equal to the same thing are also equal to one another."

<u>The five postulates.</u>

It is postulated

1) To draw a straight line from any point to any point.
2) And to produce a finite straight line continuously in a straight line.
3) And to draw a circle with any centre and radius.
4) And all right angles are equal to one another.
5) And if a straight line falling across two straight lines make internal angles less than two right angles, then the two straight lines, on the same side on which the internal angles are subtended, produced sufficiently will meet together.

The last axiom was a point of contention for mathematicians for centuries as they tried to derive it as a result from the other four axioms and failed, which eventually led to new developments in geometry. A statement considered as equivalent for the fifth postulate by the Scottish mathematician John Playfair is, "Given a line and a point outside the line, all the lines drawn through the point meet the given line except one parallel to it." The last postulate was eventually called as the parallel postulate of Euclidean geometry.

It is to be noted that a triangle can be constructed using unmarked ruler and compass with all the first three postulates and the Book1 Elements begins with the proposition to construct an equilateral triangle. As the fourth one gives the sense of orthogonality, the sum of three angles of a triangle is 180º (proposition 32, Book1, Elements), the Pythagorean theorem of right triangles and the similar triangles holding equality in ratio of corresponding sides can only be proved with the help of $5^{th}$ postulate.

[Even though Playfair's statement is considered equivalent to Euclid's $5^{th}$ postulate, the intuitions created by two statements are little bit different. In Playfair's statement, if the given line and the point are far apart and even if a non-parallel line or to say an apparent parallel to the given line passes through the given point, we may not get a better perception whether the lines will meet together after a sufficient extension of the both. But Euclid's 5th postulate makes no distinction between, if the two straight lines in which the third line crosses over lay near to each other or far apart. This means if two straight lines $L_1$ and $L_2$ are sufficiently near to each other and even each of the internal angles subtended by the third line $L_3$ on the same side of $L_3$ crossing over $L_1$ and $L_2$ is near to a right angle or one of them a right angle or a little more and that sum of both is less than two right angles, we may get an intuition that $L_1$ and $L_2$ on the same side of the subtended angles sufficiently produced will meet together. And this further makes us intuit that if a straight line $L_4$ is set far apart $L_1$ such that internal angle subtended by $L_3$ with $L_4$ is same and on the same side of the former case considered as that subtended by $L_3$ with $L_2$, then the two straight lines $L_1$ and $L_4$ on the same side of the subtended internal angles sufficiently extended will also meet together. This implied characteristic of the 5th postulate is the basis of existence of similar triangles (triangles having similar shapes with included angles same, but the corresponding sides scaled in equal ratio) in Euclidean geometry. However, the 5th postulate of Euclid gives an intuition about parallel lines, the statement may also be referred as parallel postulate.]

### 7.2. Euclid's propositions 20,22 of Book 1 Elements.

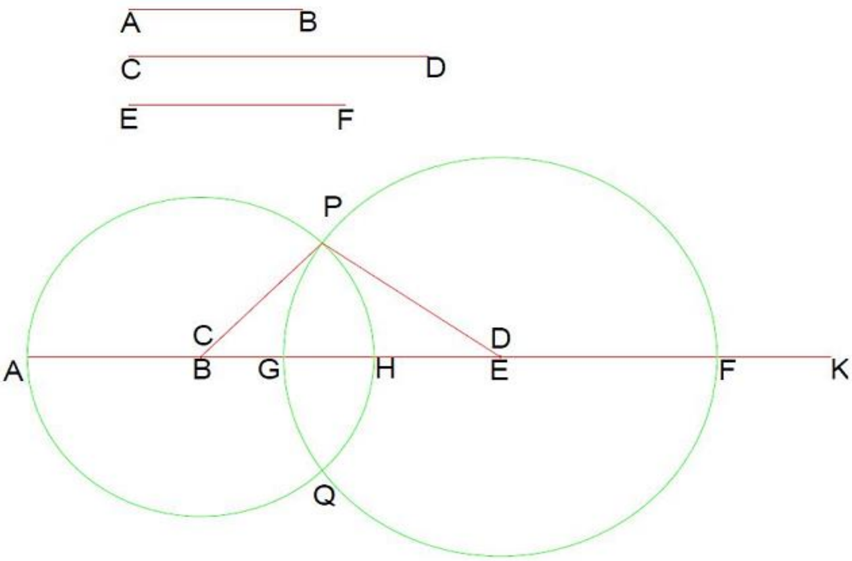
[The facts regarding Euclid's propositions throughout this paper are taken from the book of Euclid's Elements of Geometry, the Greek text written by J.L. Heiberg and translated into English by Richard Fitzpatrick. To the English author, the text in parenthesis of the propositions is material, which is implied but not actually present in Greek text.]

Euclid in his book Elements (Book 1) has stated in its proposition 20 that, "In any triangle, (the sum of) two sides taken together in any (possible way) is greater than the remaining (side)." Also, in proposition 22 of the same book it is stated that, "To construct a triangle from three straight lines which are equal to three given (straight lines). It is necessary for (the sum of) two (of the straight lines) taken together in any (possible way) to be greater than the remaining (one), (on account of the fact said in proposition 20)."

He has given proof for both the statements in the said book, as the book Elements itself is a remarkable achievement of mankind.

But if we analyse both the statements, the proposition 22 is framed in accordance with the facts established in proposition 20, which means that the proposition 20 is a necessary condition to form a triangle and if that condition is met there always exists a triangle by proposition 22. This means that to construct a triangle, "For the sum of two of the straight lines taken together in any possible way to be greater than the remaining one," is not only a necessary condition but also a sufficient condition, and the method of construction given by Euclid (for demonstrating proposition 22) itself proves that the condition is sufficient.

The construction by Euclid for demonstrating proposition 22 is explained below to show that, "For the sum of two of the straight lines taken together in any possible way to be greater than the remaining one," is also a sufficient condition to constitute a triangle.




Let the given line segments be AB, CD and EF, such that AB $<$ CD+ EF, CD $<$ AB + EF and EF $<$ AB + CD. Set out AB, CD and EF on the line segment AK as shown in the figure. Construct the first circle with B as centre and AB as radius. The condition AB $<$ CD+ EF is sufficient to ensure that the circle intersects the line segment BK somewhere only in between B and F, say at H (let H be in between B and E). Now construct the second circle with E as centre and EF as radius. Likewise, as the first circle, the condition EF $<$ AB + CD ensures that the second circle would intersect line segment AE somewhere only in between A and E. But the condition CD $<$ AB + EF is sufficient to ensure that the constructed circle will intersect line segment AE only somewhere in between A and H, say at G (otherwise, if it intersects at the point H means that CD = AB + EF or on a point in between H and E means that CD $>$ AB + EF). If H happens to be at E or in between E and F, then G will be in between

A and E. The second circle thus constructed has its diameter GF with the point G always laying inside the first circle on its diameter and the point F always laying outside the first circle on the same line segment of the diameter of the first circle extended (HK), imposes the trajectory of the second circle to intersect the first circle at points P and Q (above and below the line segment AK respectively) as shown in the figure. Joining P or Q with the centres of the circles at B and E gives the required triangle (BPE or BQE) with its sides as AB, CD and EF. It can be seen that the construction is independent of the order in which the line segments are set out initially and whether it is carried out from left to right or vice versa. Hence to construct the triangle, the given condition, two of the straight lines taken together in any possible way to be greater than the remaining one is sufficient.

[Accordingly, the construction is valid for all three-line segments $z \geq y \geq x$ (as it represents all possible logical relationships in between them) and if $z < x + y$.]

[Now, a new model, the triangle is formed from three line segments holding triangle inequality relationship between them, which has additional three parameters, the included angles between its sides.]

**7.3. Lemma 4.**

All the trilinears in natural number system formed by the relation $z \geq y \geq x$, $z < x + y$ are represented as triangles in Euclidean geometry.

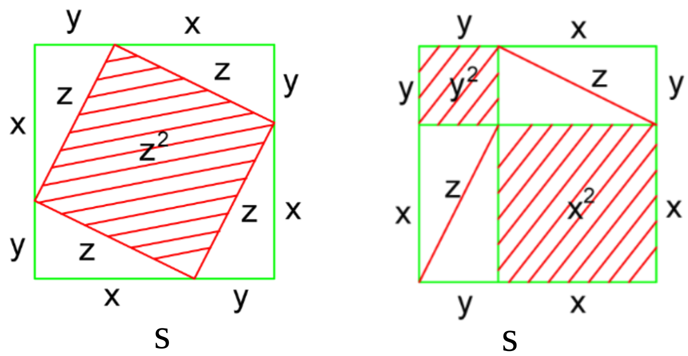
<u>Proof.</u>

The necessary and sufficient condition to form a triangle is given by propositions 20,22 of Book 1 Elements, that is the triangle inequality relationship between three-line segments. The construction of a triangle can be carried out with the unmarked ruler and compass with all the first three postulates with the given three-line segments satisfying the triangle inequality relationship.

By the linearity axiom, natural numbers can be represented as straight line segments with some scale unit for 1. The line segments constituting the trilinears are of specified magnitudes representing their corresponding numbers. But the logical relationship between these given three-line segments ($z \geq y \geq x$, $z < x + y$) to form a triangle is prime significant, and is same as the necessary and sufficient condition to construct a triangle, a triangle can always be constructed with the three-line segments. Therefore, all the trilinears in natural number system are represented as triangles in Euclidean geometry.

**7.4. The Pythagorean theorem.**

The Pythagorean theorem states that in a right-angled triangle, if z is the hypotenuse, x,y the other two sides then $z^2 = x^2 + y^2$. The proof (as believed due to Pythagoras) can be by simple reasoning (but with an intuition about space) as shown below.




On the left side we have square S with sides of length x + y containing four copies of the right-angled triangle, one in each corner, the region of S not covered by a triangle is another square of area $z^2$. On the right, the four triangles have been moved around within S to form two rectangles, the uncovered region S now consists of two squares of areas $x^2$ and $y^2$. Since moving the triangle leaves their areas unchanged, the two uncovered regions have equal areas so that $x^2 + y^2 = z^2$.

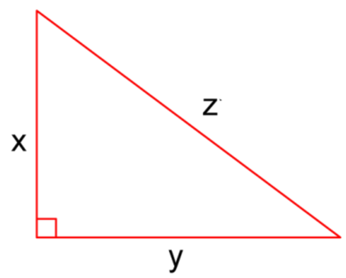
The theorem is proved with the help of the fifth postulate of Euclidean geometry (which is an axiomatic system) in proposition 47, Book 1, Elements and the theorem representing the right triangle is unique in the sense that Euclidean geometry is a consistent geometry and cannot get contradictory results from its postulates. The converse of Pythagorean theorem, "If the square on one of the sides of a triangle is equal to the sum of squares on the remaining sides, then the angle included in the remaining sides is a right angle," is stated and proved in proposition 48, Book 1, Elements.

**7.5. Lemma 5.**

The trilinear inequalities holding the relation $x^2 + y^2 = z^2$ representing the well identified trilinear class in natural number system constitute right angled triangles in Euclidean geometry, and the Pythagorean theorem of right triangles demonstrates their unique trilinear inequality to equality transformation.

<u>Proof</u>.

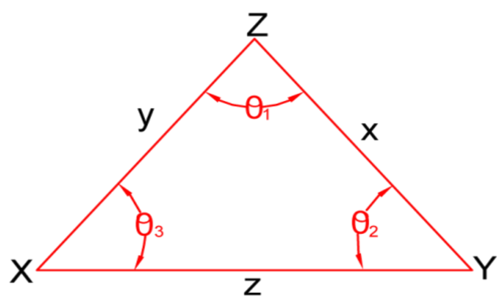
It is demonstrated in lemma 4 that all the trilinears in natural number system are represented as triangles in Euclidean geometry and include that of the well identified class of trilinears represented by the relation $x^2 + y^2 = z^2$. By proposition 48, Book 1, Elements, the triangle holding the relation $x^2 + y^2 = z^2$ should be a right triangle.




The constituted triangle represents itself the trilinear inequality between its sides and by Pythagorean theorem, relation between the sides of a right triangle is $x^2+y^2 = z^2$, and there by demonstrates their unique trilinear inequality to equality transformation.

[The above lemma establishes a comprehensive and unique correlation between natural number system and Euclidean geometry as the lemma is demonstrated with the Pythagorean theorem which is the fundamental theorem establishing the relation between sides of a triangle in Euclidean geometry and is unique. By the algebra of natural numbers, it is shown that the relationship $x^2 + y^2 = z^2$ represents a trilinear inequality in between them. Now the geometry demonstrates how this trilinear inequality looks like and why this relationship transforms in to an equality. Understanding the beauty of such relationships in nature is a marvel to human mind.]

**7.6. The cosine law – the general law of triangles in Euclidean geometry.**




As by the propositions 20 and 22 of Book I of Elements, there exists a triangle for any three-line segments satisfying the triangle inequality relationships. The triangles are classified according to the large angle, θ (large angle means, angle which can be considered as large of the three angles, i.e., if all the angles are same then the angle itself is the large angle), as obtuse angled triangle ($90º < \theta < 180º$), right angled triangle ($\theta = 90º$), acute angled triangle ($60º \leq \theta < 90º$). The cosine law generalises Pythagorean theorem for general triangles in Euclidean geometry. It represents the relationship of a side of triangle with the other two sides and the cosine of included angle between them. The equality relationship of side z with the other two sides of the triangle, x, y and the cosine of included angle ($\cos\theta_1$) between them, is $z^2 = x^2 + y^2 - 2xy\cos\theta_1$. The relationship for each of the other two sides of the triangle with the remaining sides can also be established in the same form as $y^2 = z^2 + x^2 - 2xz\cos\theta_2$ and $x^2 = z^2 + y^2 - 2zy\cos\theta_3$ where $\theta_2$ is the included angle between sides z and x and $\theta_3$, between z and y of the triangle.

In the case of obtuse angled triangle, where large angle $\theta$ (let $\theta_1 = \theta$ in triangle XYZ) is between 90º and180º, $\cos\theta$ is negative and the $- 2xy\cos\theta$ term becomes additive. Therefore, in this case, it can be shown that the relationship of the larger side z (opposite to $\theta$) with the other two sides is $x^2 + y^2 < z^2$, and in the case of right triangle, $\cos\theta = 0$ and the relationship of larger side z with other two sides is $x^2 + y^2 = z^2$. And for acute angled triangle, large angle $\theta$ is such that $60º \leq \theta < 90º$ and $\cos\theta$ takes only positive values and therefore the relationship of the larger side z with the other two sides of the triangle will be of the form $x^2 + y^2 > z^2$. The cosine law relationships of the other two sides of the triangle (other than the larger side), with their corresponding sides in each of the three cases (obtuse angled, right angled and acute angled) have no special significance, as the angle opposite to them will always be less than 90º and $\cos\theta$ will always be positive and the normal representation of cosine law will be prevalent. Therefore, a more general statement of cosine law of triangles will be $z^2 = x^2 + y^2 - 2xy\cos\theta$ where z is the larger side and $\theta$ the included angle between x and y because this relation defines the triangle completely and one could understand it as an obtuse angled, right angled or acute angled triangle. As the large angle $\theta = 180º$, $\cos\theta = -1$, the cosine law expression becomes $z^2 = x^2 + y^2 + 2xy$, in turn reduces to $x + y = z$, linear.

[The relation between sides of an obtuse angled triangle is given in proposition 12 and that of acute angled triangle is given in proposition 13 of Book 2 Elements, as statements demonstrated geometrically using Pythagorean theorem. Cosine law represents the Pythagorean theorem and both the above said relations and is a conclusive generalisation of the relation between sides of a triangle in Euclidean geometry which explicitly demonstrates that a side of a triangle related to the other two sides is also dependent on the included angle between them.]

**7.7. Lemma 6.**

All the trilinears in the natural number system which are represented as triangles in Euclidean geometry can be uniquely and invariantly expressed as $z^2 = x^2 + y^2 - 2xy\cos\theta$ where z is the large number and θ is the included angle between x and y and they are classified in to three classes according to the relations $x^2 + y^2 = z^2$, $x^2 + y^2 < z^2$, $x^2 + y^2 > z^2$.

Proof.

According to lemma 4, all the trilinears in natural number system are represented as triangles in Euclidean geometry.

All triangles in Euclidean geometry are the result of the same postulates and the Pythagorean theorem establishing the relation between three sides of a right triangle is the fundamental theorem of triangles in Euclidean geometry and is unique in this regard. The general law of triangles, the cosine law, is the generalisation of Pythagorean theorem.

The trilinears holding the relation $x^2 + y^2 = z^2$ in natural number system constitute right triangles in Euclidean geometry. The transformation of trilinear inequality to equality relation of this well identified class by Pythagorean theorem is shown in lemma 5.

As all the trilinears in natural number system are represented as triangles in Euclidean geometry, cosine law, the general law of triangles is applicable to all of them. Hence all the trilinears in natural number systems can be uniquely and invariantly transformed to the equality by cosine law as $x^2 + y^2 - 2xy\cos\theta = z^2$, where z is the larger number and θ is the included angle between x and y.

The trilinears in the natural numbers system, other than in the class of $x^2 + y^2 = z^2$ can be tentatively included in the class of $x^2 + y^2 > z^2$ or in $x^2 + y^2 < z^2$ where z is the large number. The triples forming the relationships $x^2 + y^2 > z^2$., $x^2 + y^2 < z^2$ can be uniquely represented as acute angled triangles and obtuse angled triangles respectively by cosine law of triangles. As z represents the larger number, the included angle between x and y in the case of acute angled triangles lays between 60º and 90º (inclusive of 60ºand exclusive of 90º) and in the case of obtuse angled triangles, the included angle between x and y lays between 90º and 180º (both exclusive). Cosine law validates the tentative classification of all the trilinears other than $x^2 + y^2 = z^2$ and belongs to $x^2 + y^2 < z^2$ or $x^2 + y^2 > z^2$ as genuine trilnear classification in natural number system. Therefore, for $z \geq y \geq x$, $z < x + y$, there exists only three classes of trilinears in natural number system, $x^2 + y^2 = z^2$, $x^2 + y^2 < z^2$ and $x^2 + y^2 > z^2$.

[It can be easily seen for triangles with natural number sides, from the cosine law expression $z^2 = x^2 + y^2 - 2xy\cos\theta$, the term $2xy\cos\theta$ turns out to be a negative integer in an obtuse angled triangle and to be a positive integer in an acute angled triangle and therefore $\cos\theta$ will always be a negative or positive rational number (fraction) accordingly. ]

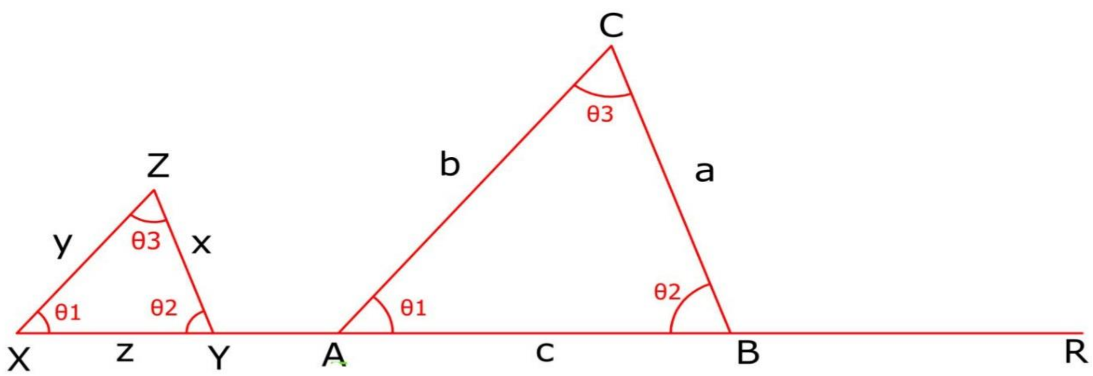
[As lemma 2 demonstrates that the trilinear triples in natural number system are countable, lemma 6 demonstrates that they are classifiable also. ]

**7.8. Euclidean geometry and the scaling property of linear and trilinear relationships in natural number system.**

It can be easily seen if the linear relationship $x + y = z$ holds between three natural numbers, then the relation $mx + my = mz$, where m is a natural number also holds true and is known as the scaling property of linear relationship in natural number system. The scaling property of linear relationship can be easily demonstrated as, a straight line segment representing mx combined with another of length my representing line segment $mx + my$ is same as arranging each of the x of line segment mx with each of the y of line segment my to form $x + y$ and all of them combined together to form $m(x + y)$ as a single line segment which is equal to mz, showing $mx + my = mz$.

The scaling property is true for trilinear relationships also as $z \geq y \geq x$, if $z < x + y$ then it follows $mz \geq my \geq mx$ and $mz < mx + my$, and a trilinear inequality relationship exists between mx, my and mz. The scaling property of trilinear relationships is demonstrated by similar triangles (triangles having similar shapes with included angles same, but the corresponding sides scaled in equal ratio) in Euclidean geometry. That is, if a triangle with sides x, y, z exists, then a scaled triangle similar to the former with sides mx, my, mz, where m is a natural number, also exists can be established with the 5$^{th}$ or parallel postulate which establishes a unique relationship between natural number system and Euclidean geometry.

<u>The scaling property of trilinears using similar triangles is demonstrated as below</u>.



Construct a triangle XYZ as shown above with line segments representing natural numbers x,y,z holding trilinear inequality. Extend the line segment XY sufficiently to R. Scale out AB in line segment YR such that AB = mz represented by the side c. Construct parallels to XZ and YZ through A and B respectively. Let the intersection of them be C. Now triangle ABC is constituted. Let AC = b and BC = a. As y parallel b and x parallel a, in triangles ABC and XYZ, angle CAB equals angle ZXY($\theta_1$), angle ABC equals angle XYZ($\theta_2$) and therefore angle ACB equals angle XZY($\theta_3$).

Now in triangle XYZ, by sine law of triangles $\frac{x}{\sin\theta_1} = \frac{y}{\sin\theta_2} = \frac{z}{\sin\theta_3}$

i.e. $x : y : z \; :: \; \sin\theta_1 : \sin\theta_2 : \sin\theta_3$

and in triangle ABC $\frac{a}{\sin\theta_1} = \frac{b}{\sin\theta_2} = \frac{c}{\sin\theta_3}$ i.e. $a : b : c \; :: \sin\theta_1 : \sin\theta_2 : \sin\theta_3$

$\therefore a : b : c :: \; x : y : z \longrightarrow \frac{a}{x} = \frac{b}{y} = \frac{c}{z}$

Since c = mz, a = mx and b = my. Therefore, the scaling property of trilinears is demonstrated by the triangle ABC, similar to triangle XYZ representing the trilinear x,y,z, with its corresponding sides scaled in equal ratio to XYZ, (mx,my,mz), is shown to exist.

[Unlike the cosine law, for a triangle with sides as natural numbers and θ is one of the included angle between its sides, can have only rational fractions for cosθ, for sine law of triangles, sinθ may turn out to be fractions with irrational numbers (incommensurables in Elements) also. The scaling property of similar triangles is demonstrated here with sine law of triangles is only due to simplicity. The results for similar triangles are geometrically demonstrated in Book 6, Elements which deals with similar figures. In proposition 4 of the Book, it is stated that, "In equiangular triangles the sides about the equal angles are proportional and those (sides) subtending equal angles correspond," and in proposition 5, it is stated that, "If two triangles have proportional sides then the triangles will be equiangular and will have the angles which subtend corresponding sides subtend equal."]

## 8. The principle of true model of relational dominance or the genesis principle of trilinear relationships in natural number system.

### 8.1 The principle of relational dominance.

For a finite collection of entities (more than two) of same characteristic with absolute magnitudes, for example, length of line segments represented by $l_1, l_2, l_3 \ldots\ldots l_n$ and if, either $l_1 = l_2 + l_3 + \ldots + l_n$, or $l_1 > l_2 + l_3 + \ldots + l_n$, $l_1$ is said to hold a dominant relation with $l_2, l_3 \ldots\ldots l_n$ and if $l_1 < l_2 + l_3 + \ldots + l_n$, $l_1$ is said to hold a dormant relation with $l_2, l_3 \ldots\ldots l_n$.

Explanation.

For any given $l_1, l_2, l_3, \ldots, l_n$, if $l_1$ has sufficient magnitude to hold $l_1 = l_2 + l_3 + \ldots + l_n$, then the relation necessarily implies that $l_1 > l_2, l_3, \ldots, l_n$, and it is implied that $l_1$ has sufficient magnitude to imply its dominance over all of $l_2, l_3, \ldots, l_n$ in the relation. If $l_1$ has more magnitude than to hold $l_1 = l_2 + l_3 + \ldots + l_n$, then it holds the relation, $l_1 > l_2 + l_3 + \ldots + l_n$.

Even though $l_1 > l_2, l_3, \ldots, l_n$, if the magnitude of $l_1$ is not sufficient enough to hold the relation $l_1 = l_2 + l_3 + \ldots + l_n$, then it holds the relation $l_1 < l_2 + l_3 + \ldots + l_n$ and the relation fails to imply $l_1 > l_2, l_3, \ldots, l_n$. Moreover, the relation $l_1 < l_2 + l_3 + \ldots + l_n$ always holds true if $l_1$ is either less than or equal to all of $l_2, l_3, \ldots, l_n$ and also if $l_1$ is greater than, equal to or less than some of them (all possible combinations in between them), means that the relation does not imply a particular condition, but in general, it is implied that $l_1$ does not have sufficient magnitude to imply its dominance over $l_2, l_3, \ldots, l_n$ in the relation.

In general, if, either $l_1 = l_2 + l_3 + \ldots + l_n$, or $l_1 > l_2 + l_3 + \ldots + l_n$, as the dominance of $l_1$ is quite evident in the relation, $l_1$ is said to hold a dominant relation with $l_2, l_3 \ldots\ldots l_n$ and since the relation $l_1 < l_2 + l_3 + \ldots + l_n$ represents the deficiency of dominance of $l_1$ over $l_2, l_3, \ldots, l_n$ in the relation, $l_1$ is said to hold a dormant relation with $l_2, l_3 \ldots\ldots l_n$.

**8.2. Lemma 7.**

For the three cases of triangles in Euclidean geometry, the obtuse angled, the right angled and the acute angled triangles, there exists a symmetry of logic between the relationship of square of large side with the sum of squares of the other two sides (established by cosine law) and the relationship of the large angle of the triangle with the sum of the other two angles.

Proof.

Let z be the large side and the other two sides be y and x ($z \geq y \geq x$, $z < x + y$) in each case (obtuse angled, right angled and acute angled triangles). Let $\theta_1$ be the large angle opposite to the large side and $\theta_2$, $\theta_3$ be the other two angles opposite to the corresponding sides y and x. The sum of the three angles of a Euclidean triangle is 180º (Proposition 32 of Book 1 Elements).

Case-1 Obtuse angled triangle.

As in this case $\theta_1 > 90^\circ$ and $\theta_2 + \theta_3 < 90^\circ$ (sum of angles has to be 180º). $\therefore \theta_1 > \theta_2 + \theta_3$.

The relationship of the larger side square with sum of squares of the other two sides is $z^2 > x^2 + y^2$ (follows the same relational logic as the angles).

Case-2 Right angled triangle.

As in this case $\theta_1 = 90^\circ$ and $\theta_2 + \theta_3 = 90^\circ$ (sum of angles has to be $180^\circ$). $\therefore \theta_1 = \theta_2 + \theta_3$.

The relationship of the larger side square with sum of squares of the other two sides is $z^2 = x^2 + y^2$ (follows the same relational logic as the angles).

Case-3 Acute angled triangle.

As in this case $\theta_1 < 90^\circ$ and $\theta_2 + \theta_3 > 90^\circ$ (sum of angles has to be $180^\circ$). $\therefore \theta_1 < \theta_2 + \theta_3$.

The relationship of the larger side square with sum of squares of the other two sides is $z^2 < x^2 + y^2$ (follows the same relational logic as the angles).

**8.3. The principle of true model of relational dominance or the genesis principle of trilinear relationships in natural number system.**

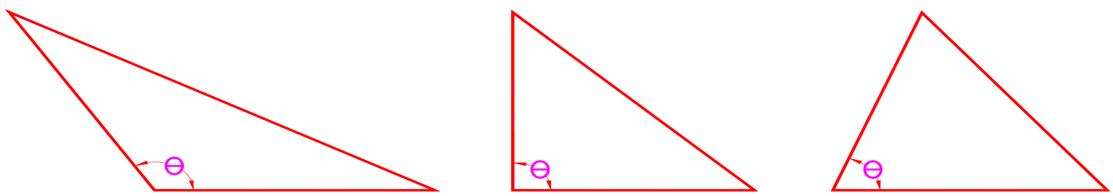
The linear relationship $x + y = z$ and the trilinear relationships, $x^2 + y^2 < z^2$, $x^2 + y^2 = z^2$ and $x^2 + y^2 > z^2$ ($z \geq y \geq x$, $z < x + y$), are the fundamental relationships of natural number system. As the natural number system follows the linearity axiom, $x + y = z$ is represented as two straight line segments x and y combined to form the straight line segment z, and the relations $x^2 + y^2 > z^2$, $x^2 + y^2 = z^2$ and $x^2 + y^2 < z^2$ are represented by the acute angled ,the right angled and the obtuse angled triangles respectively, in which the numbers x,y,z are represented by the corresponding line segments forming the linear and trilinear relations.

A new model, the triangle is formed from three line segments holding triangle inequality relationship, which has additional three parameters, the included angles between its sides. According to Euclid's proposition 18 of Book 1 Elements it has been stated and proved that "In any triangle, the greater side subtends the greater angle," and in proposition 19 of the same book it has been stated and proved that "In any triangle, the greater angle is subtended by the greater side." From both the propositions it is clear that if a specific large side exists it would subtend a specific large angle and also if a specific large angle exists in a triangle there would be a specific larger side subtending it, which implies that the largeness of angles and sides of a triangle are mutually interconnected. And by lemma 7, it can be seen that in the three cases of triangles, the acute angled, the right angled and the obtuse angled triangles, according to the logic of relational dominance of the large angle over sum of the other two angles in turn represents the logic of relational dominance of the large side (square) over (the sum of squares of) the other two sides. Regarding dominance, conversely also it is true as there exists symmetry of logic for the relation between sides of a triangle in second degree to that of angles in all the three cases.

Since the dominance characteristic of the three triangle models (trilinear structures) is determined by the concord between logic of relational dominance of angles and sides, each of them can be called as true model of relational dominance of their corresponding trilinear relationships.

The property of the obtuse angled triangle and the right-angled triangle is that there is always a predominant large angle and therefore a predominant large side opposite to that. As demonstrated in lemma 7, in the case of acute angled triangle, even if there is a specific large angle, it is not relationally dominant as in the case of right angle or more, as the larger angle is less than 90º and is always less than the sum of other two angles in the triangle. It can also be understood from the fact that the relational logic of each of the angles (even if there is a specific large angle) with the sum of other two angles for all the three cases is same for an acute angled triangle contrary to the relation of the large angle with the sum of other two angles of obtuse angled and right-angled triangles. Therefore, the larger angle in the acute angled triangle model fails to assert sufficient dominance over the other two and correspondingly the largeness of the large side (even if it is specifically large) turns out to be dormant in the relation connecting the sides.

**Large side predominant** | **Large side dominant** | **Large side dormant**



**Obtuse angled triangle** | **Right angled triangle** | **Acute angled triangle**

**Large angle** $\Theta$, $90° < \Theta < 180°$ | **Large angle** $\Theta = 90°$ | **Large angle** $\Theta$, $60° \leq \Theta < 90°$

The acute angled triangle is a model that can have two large sides and correspondingly two large angles or all the three sides and angles same as in the case of equilateral triangle. The model manifests itself with its sides as a relatively good proportioned one, more often when it has all the sides different, such that the dominance of a single side is not sharply evident contrary to obtuse angled or right-angled triangles. Thus, the relational dominance of the large side over the other two sides in the three types of triangles is also reflected in the corresponding homogeneous second degree relations representing them. In the relations $x^2 + y^2 = z^2$ and $x^2 + y^2 < z^2$, one can easily identify the large side as z ($z^2 > y^2, x^2 \Rightarrow z > y, x$). Therefore, the right angled triangle is the true model that represents a triangle inequality relationship in which the large side has its magnitude sufficient enough to imply its dominance over the other two and an obtuse angled triangle represents the more than sufficient case of it. But the relations connecting each side of acute angled triangle with other two sides are all of the same form $x^2 + y^2 > z^2$, where x and y interchange with z only in each case and as the relational logic remains same, the largeness of z (even if $z > y, x$) cannot be explicitly or implicitly understood from the relations or remains dormant unless the sides or angles are measured and specified. Therefore, it is implied that for an acute angled triangle, none of its sides has sufficient magnitude to imply its dominance over the other two and it is a true model that represents the dormant relation of any of its sides with the other two sides. In general, the triangles classify into three classes according to the sufficiency of magnitude of the large side to imply its dominance over the other two.

The above said characteristic of the triangles forms the genesis of the trilinear relations in natural number system. The trilinear triples orient themselves as obtuse angled, right angled or acute angled triangles according to the relational dominance of the large number over the other two and the relation between the sides of the triangles in each case reflects the same. It is interesting to note that the four fundamental relationships in the natural number system $x + y = z$, $x^2 + y^2 < z^2$, $x^2 + y^2 = z^2$ and $x^2 + y^2 > z^2$ can be demonstrated by the first six natural numbers i.e., 1,2,3,4,5,6, with all numbers different for each relation.

1) $1 + 2 = 3 \Rightarrow x + y = z$

$$\left.\begin{array}{ll} 2) & 2^2 + 3^2 < 4^2 \Rightarrow x^2 + y^2 < z^2 \\ 3) & 3^2 + 4^2 = 5^2 \Rightarrow x^2 + y^2 = z^2 \\ 4) & 4^2 + 5^2 > 6^2 \Rightarrow x^2 + y^2 > z^2 \end{array}\right\} \quad z > y > x,\ z < x + y$$

The "principle of true model of relational dominance" or the "genesis principle" of the trilinear relationships in natural number system may be stated as, "The trilinear triples in natural number system classify in to three different classes according to the relational dominance of the large number over the other two numbers present in them based on the three types of triangles, the obtuse angled, right angled and acute angled triangles, as they are true models of relational dominance representing them, correspondingly represented by the homogeneous second degree relations between their sides." It also shows that the natural number system which is otherwise well ordered is well structured too.

**9. Fermat's Last Theorem and its proof.**

Pierre de Fermat in 17th century stated that," No cubes of natural numbers can be split in to two cubes or a biquadrate can be split into two biquadrates or no other higher order number greater than 2 can be split in to the sum of two natural numbers having the same order. " In modern terms the theorem can be stated as $x^n+y^n = z^n$ has no solutions for $n > 2$.

Proof.

It can be seen $x^n + y^n = z^n$ is homogeneous in n and we may assume that there exists solution for the relation for all $n \geq 2$.

$$\text{Let } x^n + y^n = z^n \rightarrow (1)$$

Then $z^n > x^n$, $z^n > y^n \therefore z > x$, $z > y$. The relation between y and x is insignificant to affect the proof any way. Let one of them be equal to or greater than the other. i.e. $y \geq x$.

Now, $x^n + y^n < (x + y)^n$ ( by binomial theorem for corresponding n and $n \geq 2$).

$\therefore$ by (1) $z^n < (x + y)^n$ i.e. $z < (x + y)$

This shows that the larger number z is less than the sum of other two numbers and the other two inequalities $x < z + y$, $y < z + x$, naturally follow and it is a sufficient condition to form a trilinear. Also the scaling property of the trilinear relationship in natural number system is held true by the equation $x^n + y^n = z^n$, $n \geq 2$, as it can be easily seen if x,y,z is a solution then mx, my, mz, m a natural number, holding trilinear inequality relation among them is also a solution. This leads to a class of triples satisfying the relation for any n. Therefore, $x^n + y^n = z^n$ represents an equality relation of trilinear inequality in disguise, for all $n \geq 2$. This is a contradiction except for the case n=2.

There are infinite number of solutions for the relation $x^2 + y^2 = z^2$ representing the fundamental trilinears in the natural number system. Natural numbers follow the linearity axiom and in turn follows the general theory of relationships of straight-line segments, the Euclidean geometry. All the trilinears in the natural number system are represented as triangles in Euclidean geometry and only be represented as an equality by the general law of triangles, the cosine law, which is a unique and invariant second order relationship between the three sides of the triangle and each of them belongs to one of the following classes, $x^2 + y^2 = z^2$, $x^2 + y^2 < z^2$, $x^2 + y^2 > z^2$, z the large number, correspondingly represented by their true models of relational dominance, right angled, obtuse angled and the acute angled triangles, enunciated by Lemma 6 and the principle of true model of relational dominance of trilinear relationships in natural number system .

Now, let us analyse the case of raising the power of the triples of the three true modelled trilinear classes to higher orders and see whether they can be transformed to relations of the form $x^n + y^n = z^n$, $n \geq 3$. It is proved above that the relation represents a trilinear inequality for $n \geq 2$. And it is large number dominant for $n \geq 1$( $z^n > y^n, x^n \Rightarrow z > y, x$ ).

Consider the case of triples in acute angled triangle class holding the relation $x^2 + y^2 > z^2$

The acute angled triangle model implies that none of its sides has sufficient magnitude to imply its dominance over the other two and it is the true model representation of the large number dormant trilinear relationships in natural number system. Though the relations of the form $x^n + y^n = z^n$, $n \geq 3$, represent a trilinear inequality relationship, they also imply $z > y, x$, the dominance of the large number. Therefore, the transformation of any of the triples represented by the acute angled triangle model to any of these large number dominant trilinear relations turns out to be a clear contradiction and not possible, as the model demonstrates the relational dormancy of the large number, the relations imply the contrary. Moreover, this condition also contradicts the existence of the well-structured (true modelled) trilinear classification (the principle of true model of relational dominance of trilinear relationships) in natural number system. In other words, raising the power of the numbers representing the sides of a triangle cannot alter the concord between the relational logic of angles and sides (as shown in lemma 7) that determines the relational dominance characteristic of the triangles (trilinear structures).

Checking whether the trilinear relations $x^2 + y^2 = z^2$ and $x^2 + y^2 < z^2$ which are large number dominant represented by right angled and obtuse angled triangles respectively can hold the large number dominant trilinear relations of the form $x^n + y^n = z^n$, $n \geq 3$.

Let as consider the case of trilinear triples that belong to the class $x^2 + y^2 = z^2$. It can be shown that they cannot hold a relation of the form $x^n + y^n = z^n$, $n \geq 3$ as follows.

Multiply both sides of the equation $x^2 + y^2 = z^2$ with $z^k$, k a natural number.

Then $z^k.x^2 + z^k.y^2 = z^{k+2}$. As $z > y, x \Rightarrow z^k > y^k, x^k \Rightarrow z^k.x^2 + z^k.y^2 > x^k.x^2 + y^k.y^2 \Rightarrow z^{k+2} > x^{k+2} + y^{k+2}$. As $k = 1,2,3....$ and let $k+2 = n \Rightarrow n \geq 3$. Hence the relation $x^2 + y^2 = z^2$ can only shift to $x^n + y^n < z^n$, $n \geq 3$ when raised to powers above 2. Now the same argument can be applied to the trilinears holding the relation $x^2 + y^2 < z^2$ and shown that they also can only shift to $x^n + y^n < z^n$, $n \geq 3$ when raised to powers above 2 and therefore cannot hold the relation of the form $x^3 + y^3 = z^3$ or higher order.

Hence solutions for $x^n + y^n = z^n$, $n > 2$ do not exist and proved.

**10. Illustration of the theorem with examples.**

Let us demonstrate the above discussions, by considering the case of three triples, each from $x^2 + y^2 < z^2$, $x^2 + y^2 = z^2$, $x^2 + y^2 > z^2$ and raise it to the power of three to see what way their relational logic shifts.

$2^2 + 3^2 < 4^2 \Rightarrow x^2 + y^2 < z^2 \quad 2^3 + 3^3 < 4^3$

$3^2 + 4^2 = 5^2 \Rightarrow x^2 + y^2 = z^2 \quad 3^3 + 4^3 < 5^3$

$5^2 + 6^2 > 7^2 \Rightarrow x^2 + y^2 > z^2 \quad 5^3 + 6^3 < 7^3$

It can be seen that all of them shifts to the same relational logic when they are raised to the power of three and hence, they cannot be associated with the trilinear structures. Here in the case of $x^2 + y^2 > z^2$, when raised to higher powers above 2, it may not shift always to the form $x^n + y^n < z^n$ contrary to that of $x^2 + y^2 < z^2$ and $x^2 + y^2 = z^2$. For example, in the case of 7,8,9, $7^2 + 8^2 > 9^2$, when the triple is raised to the power of 3, $7^3 + 8^3 > 9^3$, holds the same relational logic, contrary to the case of 5,6,7. But they will never shift to the form $x^n + y^n = z^n$, $n \geq 3$, as it represents a trilinear relation that contradicts the logic of relational dominance of the trilinear structures.

Accordingly, the natural numbers holding the trilinear inequality relationship among them are no more a representation of trilinear inequality when they are raised above the power of two.

**11. Inference.**

To summarise, the natural number system obeys the linearity axiom and there exists a transformation logic, the Euclidean geometry, that uniquely transforms the trilinear inequalities in natural number system to an equality as a second-degree relationship (cosine law), thereby also demonstrates that these inequalities do not get transformed into equality relationships of the form $x^n + y^n = z^n$, $n > 2$, as it contradicts the principle of true model of relational dominance of trilinear relationships.

The natural number system, otherwise well ordered, is well structured too. It is not only the fundamental system for counting objects and combining their counts but also the fundamental system for measuring physical quantities and combining them to get resultant (the triangle law of addition of physical quantities). Euclidean geometry provides a theoretical frame work for the scheme and Fermat's Last Theorem demonstrates the system is unique.